Math 3280

① Poisson r.v. with λ: $P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$, $k=0,1,2,\dots$

$E[X]=\lambda$, $Var(X)=\lambda$.

② $E[X_1+X_2+\cdots+X_n]=\sum_{k=1}^{n}E[X_k]$

③ the distribution function of r.v. X is

$$F(x)\triangleq P(X\le x),\quad x\in\mathbb{R}.$$

Properties:
- F is nondecreasing
- F is right-continuous
- $\lim_{x\to+\infty}F(x)=1$, $\lim_{x\to-\infty}F(x)=0$
- for $a\in\mathbb{R}$, $P(X=a)=\underline{P(X\le a)}-P(X<a)$

$$=F(a)-\lim_{x\to a^-}P(X\le x)$$

$$=F(a)-\lim_{x\to a^-}F(x)$$

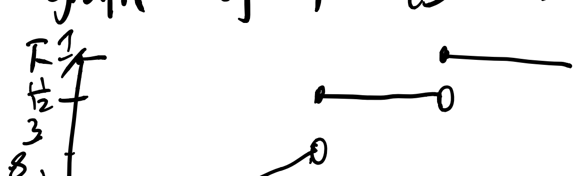
Example 1: Suppose X is a r.v. with the distribution

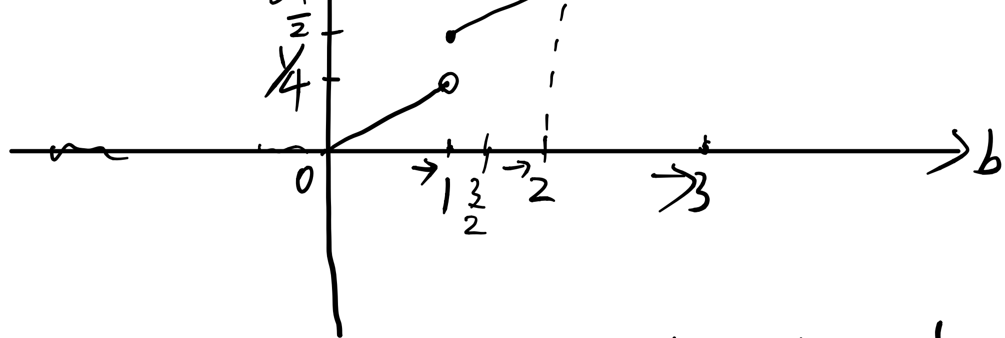
$$F(b)=\begin{cases}0 & b<0\\ b/4 & 0\le b<1\\ \underline{\frac{1}{2}+\frac{b-1}{4}} & 1\le b<2\\ 11/12 & 2\le b<3\\ 1 & 3\le b\end{cases}$$

① Find $P(X=i)$, $i=1,2,3$

② Find $P(1/2<X<3/2)$.

Solution: The graph of F is as follows:





$$P(X=1) = F(1) - \lim_{x \to 1^-} F(x) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(X=2) = F(2) - \lim_{x \to 2^-} F(x) = \frac{11}{12} - \left(\frac{1}{2} + \frac{2-1}{4}\right) = \frac{1}{6}$$

$$P(X=3) = F(3) - \lim_{x \to 3^-} F(x) = 1 - \frac{11}{12} = \frac{1}{12}$$

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \underline{P\left(X < \frac{3}{2}\right)} - P\left(X \le \frac{1}{2}\right)$$

$$= \lim_{x \to \left(\frac{3}{2}\right)^-} F(x) - F\left(\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + \frac{\frac{3}{2} - 1}{4}\right) - \frac{\frac{1}{2}}{4}$$

$$= \frac{1}{2}.$$

Example 2. Geometric r.v.

Suppose independent trials are performed until a success occurs. Each trial has a probability p to be success. X denote the number of trials in the experiment. Then X is called geometric r.v.

① Find the probability mass function of X.

② $E[X]$, $Var(X)$.

Solution: $P(X=k)=(1-p)^{k-1}\cdot p$

$k=1,2,3,\ldots\ldots$

$$E[X]=\sum_{k=1}^{\infty}k\cdot P(X=k)=\sum_{k=1}^{\infty}\overset{=(k-1)+1}{k}\cdot p\cdot(1-p)^{k-1}=\sum_{k=1}^{\infty}(k-1)\cdot p\cdot(1-p)^{k-1}+\underbrace{\sum_{k=1}^{\infty}1\cdot p\cdot(1-p)^{k-1}}_{p\cdot\frac{1}{1-(1-p)}=1}$$

$$=\sum_{k=1}^{\infty}(k-1)\cdot p\cdot(1-p)^{k-1}+1$$

$$\overset{j=k-1}{=}\sum_{j=0}^{\infty}j\cdot p\cdot(1-p)^{j-1}\cdot(1-p)+1$$

$$=(1-p)\cdot\underbrace{\sum_{j=1}^{\infty}j\cdot p\cdot(1-p)^{j-1}}_{E[X]}+1$$

$$E[X]=(1-p)\cdot E[X]+1$$

$$\Rightarrow E[X]=1/p.$$

$$\mathrm{Var}(X)=E[X^2]-(E(X))^2.$$

$$E[X^2]=\sum_{k=1}^{\infty}k^2\cdot P(X=k)=\sum_{k=1}^{\infty}\overset{=(k-1+1)^2}{k^2}\cdot p\cdot(1-p)^{k-1}$$

$$=\sum_{k=1}^{\infty}(k-1)^2\cdot p\cdot(1-p)^{k-1}+2\sum_{k=1}^{\infty}(k-1)\cdot p\cdot(1-p)^{k-1}+\underbrace{\sum_{k=1}^{\infty}1^2\cdot p\cdot(1-p)^{k-1}}_{p\cdot\frac{1}{1-(1-p)}=1}$$

$$\overset{j=k-1}{=}\sum_{j=0}^{\infty}j^2\cdot p\cdot(1-p)^{j-1}\cdot(1-p)+2\cdot\sum_{j=0}^{\infty}j\cdot p\cdot(1-p)^{j-1}\cdot(1-p)+1$$

$0^2\cdot p(1-p)^{0-1}=0.$

$$=(1-p)\cdot\sum_{j=1}^{\infty}j^2\cdot p\cdot(1-p)^{j-1}+2(1-p)\sum_{j=1}^{\infty}j\cdot p\cdot(1-p)^{j-1}$$

$E[X^2]$ $+1$ $E[X]$

$$= (1-p)\cdot E[X^2] + 2(1-p)\cdot 1/p + 1$$

$$\Rightarrow E[X^2] = \frac{2-p}{p^2}.$$

$$Var(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}.$$

Example 3. Let X be a binomial r.v. with parameters (n,p), prove that

$$E\left[\frac{1}{1+X}\right] = \frac{1-(1-p)^{n+1}}{(n+1)\cdot p}.$$

Solution:

$$E\left[\frac{1}{1+X}\right] = \sum_{k=0}^{n} \frac{1}{1+k}\cdot P(X=k)$$

$$= \sum_{k=0}^{n} \frac{1}{(1+k)}\cdot \binom{n}{k}\cdot p^k\cdot(1-p)^{n-k}$$

$$i\cdot\binom{n}{i} = n\cdot\binom{n-1}{i-1}.$$

$$\Downarrow$$

$$(k+1)\cdot\binom{n+1}{k+1} = (n+1)\cdot\binom{n}{k}.$$

$$\Rightarrow \binom{n}{k}\cdot\frac{1}{1+k} = \frac{1}{n+1}\cdot\binom{n+1}{k+1}$$

$$= \sum_{k=0}^{n} \frac{1}{(n+1)\cdot p}\binom{n+1}{k+1}\cdot p^{k+1}\cdot(1-p)^{(n+1)-(k+1)} \quad {}^{=n-k}$$

$$= \frac{1}{(n+1)\cdot p}\cdot\sum_{k=0}^{n}\binom{n+1}{k+1}\cdot p^{k+1}\cdot(1-p)^{(n+1)-(k+1)}$$

$$\overset{j=k+1}{=} \frac{1}{(n+1)p}\sum_{j=1}^{n+1}\binom{n+1}{j}\cdot p^j\cdot(1-p)^{(n+1)-j}$$

$$= \frac{1}{(n+1)p}\cdot\left(1-\binom{n+1}{0}\cdot p^0\cdot(1-p)^{(n+1)-0}\right)$$

$$= \frac{1-(1-p)^{n+1}}{(n+1)\cdot p}. \quad \checkmark.$$

4. For a non-negative integer-valued r.v. N, show that

$$\sum_{i=0}^{\infty} i \cdot P(N>i) = \frac{1}{2}\left(E[N^2] - E[N]\right).$$

Solution: $P(N>i) = \sum_{k=i+1}^{\infty} P(N=k)$

$$\sum_{i=0}^{\infty} i \cdot P(N>i) = \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} i \cdot P(N=k). \qquad k>i$$

interchange the summation order.

$$= \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} i \cdot P(N=k)$$

$$= \sum_{k=1}^{\infty} P(N=k) \cdot \underbrace{\sum_{i=0}^{k-1} i}_{=\frac{(k-1)\cdot k}{2}}$$

$$= \sum_{k=1}^{\infty} P(N=k) \cdot \frac{k^2-k}{2}$$

$$= \frac{1}{2} \underbrace{\sum_{k=1}^{\infty} k^2 \cdot P(N=k)}_{E[N^2]} - \frac{1}{2} \cdot \underbrace{\sum_{k=1}^{\infty} P(N=k) \cdot k}_{E[N]}$$

$$= \frac{1}{2} \cdot E[N^2] - \frac{1}{2} \cdot E[N].$$